Montague's Theorem and Modal Logic

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Abstract

In the present piece we defend predicate approaches to modality, that is approaches that conceive of modal notions as predicates applicable to names of sentences or propositions, against the challenges raised by Montague's theorem. Montague's theorem is often taken to show that the most intuitive modal principles lead to paradox if we conceive of the modal notion as a predicate. Following Schweizer [16] and others we show this interpretation of Montague's theorem to be unwarranted unless a further non trivial assumption is made—an assumption which should not be taken as a given.

We then move on to showing, elaborating on work of Gupta [5], Asher and Kamp [2], and Schweizer [16], that the unrestricted modal principles can be upheld within the predicate approach and that the predicate approach is an adequate approach to modality from the perspective of modal operator logic. To this end we develop a possible world semantics for multiple modal predicates and show that for a wide class of multimodal operator logics we may find a suitable class of models of the predicate approach which satisfies, modulo translation, precisely the theorems of the modal operator logic at stake.

1 Introduction

Prima facie there seem to be two options how one can go about when formalizing modal notions. One is the now common treatment of modalities as sentential operators, that is

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modalities are conceived as expressions that take a sentence as argument and yield a new sentence. The other so-called predicate (or syntactical) approach conceives of modalities as predicates applicable to names of sentences or propositions. Even though the predicate approach to modality seems to be a perfectly viable alternative to the standard operator approach it has received little attention despite prominent advocates like Carnap and Quine. This neglect may be due to the impressive story of mathematical success of modal operator logic which originated with the development of possible world semantics by Kanger, Kripke, Hintikka and others. Another reason for this underrepresentation of predicate approaches to modality, however, might be attributed to what is nowadays called Montague's theorem and which is commonly interpreted as showing that the intuitive and constitutive modal principles lead straight to paradox once we take the modality at stake to be aptly formalized by a predicate. For example, Slater [18] even takes Montague's theorem to show that we have to give up predicate approaches to modality, which he calls syntactic approaches, altogether:

> "Since Montague, we surely now know that syntactic treatments of modality must be replaced by operator formulations." (Slater [18], p. 453)[1]

In this piece, following up on the work of Niemi [14], Gupta [5], Asher and Kamp [2], and Schweizer [16], we take issue with this interpretation of Montague's theorem. On the one hand, if this interpretation of Montague's result were correct, Tarski's undefinability result would establish that truth ought to be treated as an operator. For Tarski's result would then establish that we cannot unrestrictedly adopt the most constitutive and intuitive principle of the notion of truth, namely Tarski's convention $\mathcal{T}$, for sake of paradox. Yet, virtually no-one has taken Tarski's undefinability theorem to show that truth ought not be treated as a predicate.[2] On the other hand, or so we argue, we may consistently adopt the principles of modal operator logic in the predicate setting, that is we point out that Montague's result hinges on a further non trivial condition and show the consistency and adequacy of a wide class of modal logics in the predicate setting. Indeed, we even provide consistency and adequacy results for a wide range of multimodal logics.

Accordingly, the plan of the paper is the following. We start by a careful discussion of Montague's theorem and argue that his result hinges on the fact that the names of the sentences we use in stating the modal principles are what we call their arithmetical names (or something alike). Next we show how Gupta [5] used this observation in the case of truth to construct a classical model in which the unrestricted convention $\mathcal{T}$ would obtain. A fact one would initially have taken to contradict Tarski's undefinability result. We then go on to

[1] Montague [11] and Kripke [10] comment in the same vein although their formulation are much more careful.
[2] See Skyrms [17] for remarks along these lines.

showing how Gupta's construction may be used to construct a possible world semantics for modalities conceived as predicates. A semantics in which the unrestricted modal principles may be satisfied. Finally, in the last section of this paper we turn toward adequacy results of the kind proposed by Skyrms [17] and Schweizer [16]. That is, we show that for a wide class of multimodal operator logics we may find a suitable class of models of the predicate approach that satisfy modulo translation precisely the theorems of the modal operator logic at stake. More precisely, we show that if a modal operator logic is complete with respect to a class of frames of possible world semantics for modal operator logic in which the accessibility relation has a certain property Φ, then modulo translation all theorems of the modal operator logic under consideration will be true in a class of possible world frames of the predicate approach in which the accessibility relation equally has property Φ. This result may be taken to suggest that the predicate approach to modality is adequate from the perspective of modal operator logic. The novelty our work resides in the generalization of Gupta's work to a possible world semantics for multiple modalities[3] and, most importantly in the generalization of the mentioned adequacy results à la Schweizer [16] to the broad class of multimodal logics hinted at above. Moreover, the strategy we use in establishing this latter result is genuine and departs from the strategy used in Schweizer [16]. We now start with the discussion of Montague's theorem.

2 Montague's Theorem

As we have just mentioned we will now introduce and discuss Montague's theorem. To this end we highlight the necessary background assumption of Montague's theorem, discuss its consequences and, eventually, point toward a possible way out of the dilemma posed by Montague's result for the proponent of the predicate approach to modality.

It is well known that within relatively weak arithmetical theories, e.g. Robinson arithmetic Q, we may encode syntax by the method of Gödel numbering and thereby provide names for all expressions of the language under consideration. The name of an expression ζ would then be the numeral of the Gödel number of ζ. In what is to come we shall denote this "arithmetical name" of ζ by $\ulcorner\zeta\urcorner$. Yet, if we can encode syntax in this way, we may prove the so-called diagonal lemma which in its simplest form tells us that for any formula $\phi(x)$ where x is the only free variable in ϕ we may find a sentence δ that is provably equivalent to $\phi(\ulcorner\delta\urcorner/x)$.[4] The diagonal lemma thus allows us to find sentences that—in a certain sense—

[3] Asher and Kamp [2] equally provide a possible world semantics for the predicate setting but their framework is slightly different and only allows for one modal notion.

[4] A sentence is a formula with no free variable.

make assertions about themselves and thereby introduce self-reference into the framework. In particular, if we have a truth or modal predicate in our language we may find a sentence which is provably equivalent to the sentence which asserts that it is not true (or not necessary) using the arithmetical name of the initial sentence for this claim. Now, Montague exploited the existence of this kind of sentences to show that any theory Σ in which Q can be relatively interpreted and where a formula $\alpha(x)$ obeys the modal principles

$$(T_\alpha) \qquad \alpha(\ulcorner\phi\urcorner) \to \phi$$

$$(Nec_\alpha) \qquad \frac{\phi}{\alpha(\ulcorner\phi\urcorner)}$$

is inconsistent. Now, these two principles seem, at least intuitively, impeccable for most modal notions and thus Montague's theorem is thought to put a considerable amount of pressure on predicate treatments of modality. For example, if we consider the notion of necessity then (T) asserts that what is necessary is the case where (Nec) asserts that when have inferred (or proved) ϕ we may conclude to its necessity.

For expository ease we state Montague's theorem in a slightly less general way and omit the complication of relative interpretability. Instead we require a theory Σ to be an extension of Q in a language of arithmetic $\mathcal{L}$ or an extension thereof. But we start by presenting Tarski's undefinability theorem for this will allow us to view Montague's theorem as a straightforward strengthening of the former theorem

Theorem 2.1 (Tarski/Gödel)**.** *Let Σ be a theory extending Q in $\mathcal{L}$ and α a (possibly complex) one-place predicate. If for every sentence $\phi \in \mathcal{L}$*

$$(i) \qquad \Sigma \vdash \alpha(\ulcorner\phi\urcorner) \leftrightarrow \phi$$

then Σ is inconsistent.

Proof. As a consequence of the diagonal lemma there is a sentence λ such that

$$\Sigma \vdash \lambda \leftrightarrow \neg\alpha(\ulcorner\lambda\urcorner)$$

but since by (i) $\Sigma \vdash \alpha(\ulcorner\lambda\urcorner) \leftrightarrow \lambda$ the contradiction is immediate. □

As we have already pointed out, Montague's theorem can now be presented as a strengthening of the above theorem. That is, one direction of (i)—of course, (i) asserts that Σ proves the Tarski biconditionals (TB)—can be replaced by a rule of inference for the inconsistency to still obtain. The inconsistency can then be derived using the sentence λ we employed in the proof of theorem 2.1.

Theorem 2.2 (Montague)**.** *Let* Σ *a theory extending* Q *in* $\mathcal{L}$ *and* α *a (possibly complex) one-place predicate. If for every sentence* $\phi \in \mathcal{L}$

(*i*) $$\Sigma \vdash \alpha(\ulcorner \phi \urcorner) \to \phi$$

(*ii*) $$\Sigma \vdash \phi \Rightarrow \Sigma \vdash \alpha(\ulcorner \phi \urcorner)$$

then Σ *is inconsistent.*

Proof. Again the proof makes essential use of the diagonal lemma:

1. $\Sigma \vdash \neg\alpha(\ulcorner \lambda \urcorner) \leftrightarrow \lambda$ — Diagonal lemma
2. $\Sigma \vdash \alpha(\ulcorner \lambda \urcorner) \to \lambda$ — (*i*)
3. $\Sigma \vdash \neg\alpha(\ulcorner \lambda \urcorner)$ — 1, 2
4. $\Sigma \vdash \lambda$ — 1, 3
5. $\Sigma \vdash \alpha(\ulcorner \lambda \urcorner)$ — 4, (*ii*)

□

Clearly, Tarski's and Montague's inconsistency result depend on the modal principles characterizing α and, more importantly, on the possibility of finding a sentence λ which says of itself that it is not α and thus on the ability to diagonalize on the predicate α. If one were to block the possibility of diagonalizing α, the paradox could no longer be derived along the outlines of theorems 2.1 and 2.2. Basically, this is the strategy of Tarski who, by typing the truth predicate, did not allow for names of sentences in which the truth predicate occurs in the argument position of the truth predicate itself. To appreciate how this blocks the derivation of the paradox notice that the sentence λ is in more explicit terms the sentence

$$\neg\alpha(sub^{\bullet}(\ulcorner \neg\alpha(sub^{\bullet}(v_0, v_0))\urcorner, \ulcorner \neg\alpha(sub^{\bullet}(v_0, v_0))\urcorner))$$

and therefore λ cannot be constructed, if iterations of α are prohibited by the formation rules of the language.[5]

While this strategy is successful in blocking the paradoxes it is inherently *ad hoc* in flavor. After all, how could imposing such syntactic restrictions on the formation rules of the language be justified on philosophical grounds? And at least *prima facie* it seems that this *ad hoc* flavor transfers to all strategies that resolve the paradoxes by blocking the diagonalization of the truth or the modal predicate. Moreover, several central modal principles involve

[5]Here, $sub^{\bullet}(\cdot, \cdot)$ represents the binary substitution function that takes the Gödel number of a formula ϕ with exactly one free variable and the Gödel number of some expression ζ of the language as arguments and provides the Gödel number of the sentence that results from ϕ when the free variable is replaced by the numeral of the Gödel number of ζ as an output.

iterated modalities which we cannot represent—at least in a straightforward manner—if we opt for a typing solution. Accordingly, from this perspective it seems the proponent of the operator approach has got a point in claiming that predicate approaches to modality fail to provide an adequate treatment of the modal notions.

However, at least at the outset there does not seem to be any philosophical justification or requirement for the names of the sentences to be the numerals of the Gödel codes of the sentences, i.e. the arithmetical names, as assumed in Montague's theorem. And this fact has been exploited by e.g. Niemi [14], Gupta [5] and Schweizer [16].[6] Gupta, for instance, shows that once we postulate a distinct class of quotation names which are used to state the Tarski biconditionals (TB), we can construct a classical model for truth that meets, contra Tarski, an unrestricted form of convention $\mathcal{T}$. As we shall see Gupta's strategy generalizes to the modal case and can even be used to provide a predicate account of multiple modalities.

Gupta's construction relies on an observation which was probably noted explicitly for the first time by Niemi [14] who provided a predicate account of modalities embracing, amongst other principles, the modal principles (T) and (Nec). Remember that according to Montague's theorem these principles lead to an outright contradiction in the predicate setting. But Niemi showed the consistency of his modal theory which he achieved by stipulating a distinct class of names for the sentences of the language and using these names to state his modal axioms. Contrary to the arithmetical names which serve as the names of the sentences in Montague's formulation of the modal principles these newly introduced names of sentences do not necessarily provide the resources to represent syntax, i.e. concepts of syntax such as the substitution function. But these resources are of crucial importance in the derivation of the modal antinomies or the undefinability of truth.

This reveals a further often unnoticed assumption the derivation of Tarski's and Montague's theorem relies on which is nicely summed up by Schweizer:

> "There are really two independent assumptions built into the above phenomenon of modal self-reference:
>
> (a) the possession of a class of terms structurally rich enough to do arithmetic and to sustain the diagonal lemma, and
> (b) the use of these terms as the privileged names of syntactical objects in defining the modal logic." (Schweizer [16], p. 7)

Let us observe how the derivation of Tarski's theorem can be blocked if assumption (b) is dropped. As language we assume some arithmetical language $\mathcal{L}$ supplemented by a class

[6] In a way this had also been the strategy of Skyrms [17] although Skyrms equally blocks the diagonalization of the modal predicate by syntactic means.

of quotation names for the sentences of the language and a truth predicate. These quotation names are conveyed by formulas flanked by squiggly quotes, i.e. $\lceil\phi\rceil$ is the name of the sentence ϕ. Furthermore we shall assume a theory Σ extending Q that proves the following version of (TB)

$$(TB) \qquad T\lceil\phi\rceil \leftrightarrow \phi$$

Our theory Σ still has the means of proving the diagonal lemma and thus we can find a sentence λ for which Σ proves

$$(L) \qquad \neg T\ulcorner\lambda\urcorner \leftrightarrow \lambda$$

But this by no means licenses us to derive a contradiction, given the way (TB) was stated. We can only derive

$$\neg T\ulcorner\lambda\urcorner \leftrightarrow T\lceil\lambda\rceil$$

In order to derive a contradiction we would need to substitute $\ulcorner\lambda\urcorner$ for $\lceil\lambda\rceil$ above or conversely. But such a substitution will only be licensed if

$$(GQ) \qquad \ulcorner\lambda\urcorner = \lceil\lambda\rceil$$

and this identity statement should by no means taken as a given. Rather one motivation for introducing a class of quotation names might be to block the common identification of sentences with their codes, i.e. their Gödel numbers. In this case (*GQ*) should not be expected to be provable in a theory Σ nor should one expect (*GQ*) to be a true statement since—as mentioned—the two terms will refer to two different objects. Alternatively, one might suspect that a statement parallel to (*L*) but where the occurrence of the code of λ is substituted by the quotation name of λ, i.e.

$$(QL) \qquad \neg T\lceil\lambda\rceil \leftrightarrow \lambda$$

can be proven within Σ. Yet, as we have pointed out, one should not expect (*QL*) to be trivially provable in Σ since contrary to the case of the arithmetical terms we do not know whether our quotation names "are structurally rich enough to do arithmetics and to sustain the diagonal lemma". Accordingly, one might introduce quotation names in order to work within a setting in which names for the sentences of the language are available without being prone to the effects of diagonalization. In fact this is precisely the framework of the accounts of Niemi, Gupta and Schweizer. Now, before we turn to approaches of truth and the modalities which are built upon this idea of postulating a distinct class of quotation names let us briefly summarize our findings.

Clearly, Montague's theorem puts pressure on predicate approaches to modality for it shows that we may not naively adopt the most basic modal principles we know from modal operator logic. However, one should not take his result to be conclusive in showing that predicate approaches to modality are to be replaced by operator formulations as Slater would have it. Rather, if this were the right conclusion, we would also need to treat truth as an operator. But this is a conclusion virtually nobody has drawn from Tarski's undefinability theorem. Yet we have seen that there is further respect in which Montague's result fails to be conclusive and which is more important to our inquiry: there seems to be no obvious argument why we ought to state the modal axioms using the arithmetical names instead of some alternative quotation names.

3 A Classical Model for Truth

In this section we show how Gupta [5] made use of a distinct class of quotation names to construct a classical model for truth which satisfies an unrestricted version of Tarski's convention. Gupta's construction will prove useful when we construe possible world semantics for modal predicates.

In contrast to e.g. Niemi, Gupta does not construct a particular theory but shows that independently of the base language and theory assumed we can construct a classical model for truth in which the principle (TB) holds unrestrictedly. Gupta's basic strategy is to start with a model in which the truth predicate is assigned an arbitrary extension and then revise this extension in a sequence of steps, i.e. the extension of the truth predicate at step $\alpha + 1$ is the set of sentences true in the model at step α. At limit ordinals all sentences which have remained stably in the extension of the truth predicate from an ordinal $\beta < \alpha$ onward are gathered to built the extension of T at the limit ordinal. We know that this process will not in general lead to a classical model of the language in which (TB) holds unrestrictedly, if no further assumption is made on behalf of the interpretation of the quotation names, that is the names we use to state the principles of truth, is made. It needs to be guaranteed that

- the denotatum of the quotation name of a sentence is not the denotatum of the arithmetical name of the sentence—its Gödel number for instance—and
- no function symbol or predicate is interpreted as a function or relation on the denotata of the quotation names in a way that allows us to interpret a sentence of the language as the liar sentence or some related paradoxical sentence.

Correspondingly, Gupta shows that any initial model in which these two conditions are met can be extended to a model in which (TB) holds unrestrictedly. That is, Gupta shows that in this case (all) the sequence(s) of revisions of the interpretation of the truth predicate converge(s) to a unique fixed point at the first limit ordinal ω.

While the first condition can be imposed on the interpretation function in a straightforward manner, it is unclear what we need to require exactly of the interpretation in order for the latter condition to be satisfied. It is clear, however, that if we do not allow the interpretation of predicates and function symbols other than the truth predicate to discriminate between sentences, then it will be satisfied. This condition is clearly too strong and can be liberalized in different ways. The interpretation of certain predicates can discriminate between sentences. For instance, we could allow for a predicate saying of a sentence that it is the negation of another sentence. Similarly, predicates for the remaining truth functional connectives could be introduced without harm. However, to our knowledge, it remains an open question how to spell out the second condition in its most liberal fashion, whilst guaranteeing that the revision sequences still converge to a unique fixed point.

We shall sketch Gupta's construction assuming a very restrictive condition on the interpretation function of models.[7] We start by defining the language $\mathcal{L}_{QT}$ which has the peculiarity of possessing besides the usual terms and predicates a class of quotation names and a truth predicate. The vocabulary consists of the symbols of an arbitrary denumerable first-order language augmented by the quotation symbols '$\int$' and '$\wr$' and the truth predicate 'T'. As one would suspect, in the presence of quotation names the expressions 'formula' and 'term' are defined by a simultaneous induction.

Definition 3.1 (Term, Formula and Quotation degree)**.** *The expressions 'term', 'formula' and 'quotation degree' of $\mathcal{L}_{QT}$ are defined simultaneously. The quotation degree is a function qd : $Term_{\mathcal{L}_{QT}} \cup Frml_{\mathcal{L}_{QT}} \longrightarrow \omega$ assessing the depth of embeddings of quotations in a formula:*[8]

1. *If t is a variable or an individual constant which is not a quotation name, then t is a term of $\mathcal{L}_{QT}$ with $qd(t) = 0$;*
2. *if $t_1, \dots, t_n$ are terms of $\mathcal{L}_{QT}$ and f^n is a function symbol, then $f^n(t_1 \dots t_n)$ is a term of $\mathcal{L}_{QT}$ with $qd(f^n(t_1 \dots t_n)) = max(qd(t_1), \dots, qd(t_n))$;*
3. *if $t_1, \dots, t_n$ are terms of $\mathcal{L}_{QT}$ and P^n a n-ary predicate constant, then $P^n t_1, \dots, t_n$ is formula of L with $qd(P^n t_1, \dots, t_n) = max(qd(t_1), \dots, qd(t_n))$;*
4. *if ϕ is a formula, then $\neg\phi$ and $\forall x\phi$ are formulas with $qd(\neg\phi) = qd(\forall x\phi) = qd(\phi)$;*
5. *if ϕ and ψ are formulas, then $\phi \wedge \psi$ is a formula with $qd(\phi \wedge \psi) = max(qd(\phi), qd(\psi))$;*
6. *if ϕ is a sentence of $\mathcal{L}_{QT}$, then $\int\phi\wr$ is a term of $\mathcal{L}_{QT}$ with $qd(\int\phi\wr) = qd(\phi) + 1$.*

[7] Our presentation closely follows Gupta [5], pp. 9-15.

[8] '$Term_{\mathcal{L}_{QT}}$' and '$Frml_{\mathcal{L}_{QT}}$' denote the set of terms and, respectively, of formulas of the language $\mathcal{L}_{QT}$. Similarly, '$Sent_{\mathcal{L}_{QT}}$' stands for the set of sentences of $\mathcal{L}_{QT}$.

Only certain models can be extended to a classical model satisfying (TB). We call models that qualify in this respect proper premodels.

Definition 3.2 (Proper premodel)**.** *A proper premodel of $\mathcal{L}_{QT}$ is a tuple (D, I) with $Sent_{\mathcal{L}_{QT}} \subset D$ and an interpretation function I on the whole vocabulary of $\mathcal{L}_{QT}$ except the truth predicate such that I has the following properties:*

1. *$I(\ulcorner\phi\urcorner) = \phi$ for every sentence ϕ;*
2. *if a term t is not a quotation name, then $I(t) \notin \mathcal{L}_{QT}$;*
3. *if P is an n-place predicate and $d_i \in Sent_{\mathcal{L}_{QT}}$ for $1 \leq i \leq n$, then $(d_1, \ldots, d_i, \ldots d_n) \in I(P)$ iff for all $d'_i \in \mathcal{L}_{QT}$, $(d_1, \ldots, d'_i, \ldots d_n) \in I(P)$.*
4. *No sentence appears in the range of $I(f)$, f being an n-place function symbol. If $d_i, d'_i \in Sent_{\mathcal{L}_{QT}}$, $1 \leq i \leq n$, then $I(f)(d_1, \ldots, d_i, \ldots, d_n) = I(f)(d_1, \ldots, d'_i, \ldots, d_n)$.*

We denote the class of all proper premodels by $\mathfrak{M}$.

The definition 3.2 reflects the informal condition set out before. Item 1 and 2 guarantee that the quotation name of a sentence and its arithmetical name will not refer to the same object, i.e. we distinguish between the sentence and its code. Items 3 and 4 on the other hand guarantee that the second condition is met. The interpretation of predicates other than 'T' does not discriminate between different sentences. Either no sentence appears as a relatum in the interpretation of a predicate or all of them do. The same holds for the interpretation of a function symbol but, additionally, no sentence is allowed in the range of a function interpreting a function symbol. The definition guarantees that no sentence is viewed as the liar sentence or some other paradoxical sentence. Intuitively, this is the reason why we can transform a proper premodel into a model in which (TB) is satisfied.

By definition 3.2 a premodel of the language $\mathcal{L}_{QT}$ is a tuple (D, I) which as of yet does not assign an interpretation to the truth predicate. To obtain a full-fledged model of the language $\mathcal{L}_{QT}$ we need to provide some interpretation of the truth predicate. Such a full fledged-model for the language $\mathcal{L}_{QT}$ is denoted by (M, S) where M is a proper premodel and S, an arbitrary subset of the domain, serves as the interpretation of the truth predicate. The idea is to start with some arbitrary subset of the domain and then, through a series of revisions, to obtain an interpretation of the truth predicate with some desirable properties. To this end we shall define a classical jump or revision operation Ξ_M on the domain of a proper premodel M.

Definition 3.3 (Jump relative to a proper premodel)**.** *Let M be a proper premodel with domain D. Then $\Xi_M : P(D) \longrightarrow P(D)$ is a jump operation relative to M iff for all $S \subseteq D$*

$$\Xi_M(S) := \{\phi \in Sent_{\mathcal{L}_{QT}} : (M, S) \models \phi\}$$

We may iterate applications of Ξ_M to a given set $S \subseteq \omega$ and thereby obtain a series of revisions of the interpretations of the truth predicate. We define by transfinite recursion:

Definition 3.4. *Let M be a proper premodel, Ξ_M a jump relative to M, and $S \subseteq D$. Then we set for ordinals α*

$$\Xi_M^\alpha(S) := \begin{cases} S & \text{, if } \alpha = 0 \\ \Xi_M(\Xi_M^\delta(S)) & \text{, if } \alpha = \delta + 1 \\ \{\phi \in Sent_{\mathcal{L}_{QT}} : \exists\kappa(\phi \in \bigcap_{\kappa\leq\beta<\alpha} \Xi_M^\beta(S))\} & \text{, if } \alpha \text{ is a limit ordinal} \end{cases}$$

As mentioned, it can be shown that this process reaches a unique fixed-point at the first limit ordinal ω. The following lemma is of crucial importance for establishing this fact.

Lemma 3.5. *Let M be a proper premodel with domain D. Then for all $S, S' \subseteq D$, all natural numbers n, and all ordinals α, if $\alpha > n + 1$, then for all sentences ϕ with $dq(\phi) \leq n$:*

$$\phi \in \Xi_M^{n+2}(S) \Leftrightarrow \phi \in \Xi_M^\alpha(S')$$

The lemma establishes that relative to a proper premodel the truth of a sentence of quotation degree n is settled latest at stage $n+2$. From this point on a sentence is either stably in the extension of the truth predicate or it is stably not in the extension independently of the choice of the initial interpretation of the truth predicate.

Sketch of a proof. Gupta [5] (pp. 9-15) gives a detailed proof of the lemma. We will confine ourselves to giving the crucial ideas of the proof. The proof works by induction over n and uses a side induction on α. Furthermore, three cases can be distinguished with respect to α. Either (i) α is zero, or (ii) α is a limit ordinal, or (iii) α is a successor ordinal $\delta + 1$. The first two cases are trivial—(ii) due to definition 3.4 and the induction hypothesis (induction on α).

With respect to case (iii) one can observe that by definition 3.4 $\Xi_M^{n+2}(S)$ and $\Xi_M^{\delta+1}(S')$ coincide on the sentences ϕ of quotation degree $\leq n$ iff

$$(M, \Xi_M^{n+1}(S)) \models \phi \Leftrightarrow (M, \Xi_M^\delta(S')) \models \phi$$

By induction hypothesis (induction on n) we know that the two models agree on the sentences of quotation degree $< n$. On the other hand the set $\Xi_M^{n+1}(S)$ partitions the sentences of degree $\geq n$ in two distinct denumerable sets. This is due to the fact that $(M, \Xi_M^{n+1}(S))$ is a classical model and thus there are denumerable many tautologies and denumerable many contradictions of degree $\geq n$. The same holds for $(M, \Xi_M^\delta(S'))$. This allows us to define a bijection σ from the domain onto itself which respects the interpretation of the truth predicate, i.e.

$$\phi \in \Xi_M^{n+1}(S) \Leftrightarrow \sigma(\phi) \in \Xi_M^\delta(S')$$

where σ is the identity function on all the members of the domain which are not sentences of degree $\geq n$. In virtue of definition 3.2 and the fact that any quotation name of quotation degree $\leq n$ has a sentence of degree $< n$ as its denotatum we can establish for any formula $\phi(x_1, \ldots, x_n)$ of quotation degree $\leq n$ and $d_1, \ldots, d_n \in D$

$$(M, \Xi_M^{n+1}(S)) \models \phi[d_1, \ldots, d_n] \Leftrightarrow (M, \Xi_M^{\delta}(S')) \models \phi[\sigma(d_1) \ldots, \sigma(d_n)]$$

But then as a corollary $(M, \Xi_M^{n+1}(S))$ and $(M, \Xi_M^{\delta}(S'))$ agree on the sentences of quotation degree $\leq n$ and consequently so do $\Xi_M^{n+2}(S)$ and $\Xi_M^{\alpha}(S')$.[9] □

Using lemma 3.5 we can establish our main claim, namely that we reach a fixed-point at the first limit ordinal ω, which implies that for any premodel M with domain D and $S \subseteq D$ the $\mathcal{L}_{QT}$-model $(M, \Xi_M^{\omega}(S))$ satisfies (TB):

Theorem 3.6. *For any $S \subseteq D$ and any premodel M and all sentences ϕ*

(*i*) $$\phi \in \Xi_M^{\omega}(S) \Leftrightarrow (M, \Xi_M^{\omega}(S)) \models \phi$$

(*ii*) $$(M, \Xi_M^{\omega}(S)) \models T\langle\phi\rangle \leftrightarrow \phi$$

Theorem 3.6 may be read as establishing that, contra Tarski, convention $\mathcal{T}$ can be satisfied in a semantically closed language.

At this point we do not wish to evaluate whether this construction might serve as a viable conception of truth in its own right. However, what seems important to realize is that the construction can be generalized in order to rebut Montague's assessment that virtually all of modal logic must be sacrificed, if we treat modalities syntactically[10]—at least, if this assessment is understood in its straightforward, general way.

4 Models for Modalities Conceived as Predicates

We shall now generalize Gupta's account and introduce an arbitrary finite number of modal notions into the picture. To this end we combine the construction we have just sketched with possible world semantics as known from modal operator logic. As a matter of fact we shall later exploit the similarity between possible world semantics for modal operator logic and the semantics to be constructed to show that consistent accounts of modality are not only possible but adequate, if modal operator logic is taken to be an adequate account of the modalities as is widely believed. In this we follow the lead of Skyrms [17] and Schweizer

[9] σ needs to respect the interpretation T as, e.g., a sentence of the form $\forall x Tx$ is of quotation degree 0.

[10] Cf. Montague [11], p. 294. Page numbers refer to the reprint in Montague [12].

[16] who have already established—using similar techniques—the adequacy of syntactical modal logic for a single modality with respect to the modal operator logic $S5$.

Extending the approach to multiple modalities is interesting from two perspectives. First, natural language incorporates multiple modalities and thus from this perspective a formal approach allowing for the joint treatment of several modal notions seems to be asked for. But second, Niebergall [13], Halbach [6, 7] and, Horsten and Leitgeb [8] have shown that new inconsistency results might arise, if we allow for multiple, interacting modal predicates.

The basic idea of the present proposal is to construct a possible world semantics by quantifying over "Gupta-style" models. Consequently we will evaluate a modal predicate with respect to a class of "accessible" models. More precisely, the interpretation of a modal predicate will be the intersection of the interpretations of the truth predicates in the models accessible from the present model.

In "A Syntactical Approach to Modality" Schweizer [16] puts forward a similar approach as he also quantifies over "Gupta-style" models. But his approach is less general in two respects. First, Schweizer considers only one modality and therefore does not provide an account that can deal with multiple modalities, but second Schweizer only considers the modal logic S5 and thus does not allow the accessible models to vary from one model to another.[11]

A related account has been developed by Asher and Kamp [1, 2] who also construct a possible world semantics based on Gupta's and Herzberger's Revision Theory of Truth. The inquiry of Asher and Kamp [2] is probably closest to the present undertaking, however they work in a slightly different—and to our mind slightly more complicated—-setting. In their work they only consider one modal predicate, but a generalization to multiple modal predicates seems to be rather straightforward.

We will work in a language $\mathcal{L}_{QM}$ which is like $\mathcal{L}_{QT}$ except that we add a finite number of one place modal predicates, say $N_1, \ldots, N_n$. The expressions 'term', 'formula', and 'quotation degree' are defined by the ovious modification of definition 3.1. We do not reproduce the definition. Accordingly our language possesses quotation names for all the sentences of the language. We may also adopt definition 3.2 without change (obviously the modal predicates may discriminate between different sentences).

Contrary to the case of a single truth predicate that we discussed above, the interpretation of truth and the modal predicates will not simply be defined relative to one model but rather

[11] In possible world semantics for modal operator logic the logic S5 characterizes a frame based on an equivalence relation. It can be shown that in this case accessibility relation can be dropped and instead of quantification *simpliciter* over a set of possible worlds simpliciter. Cf. Hughes and Cresswell [9] and Blackburn et al. [3] for more on modal logic and possible world semantics.

the interpretations will depend upon a modal frame. A modal frame consists of a nonempty set of premodels $\mathcal{W}$ which will serve as our set of possible worlds and a finite number of accessibility relations defined on this set. Besides the notion of a modal premodel frame we need the notion of an evaluation. An evaluation assigns to every world, i.e. every premodel in $\mathcal{W}$, a set of sentences of $\mathcal{L}_{QM}$ which will serve as the interpretation of the truth predicate at this world:[12]

Definition 4.1 (Modal Premodel Frame, evaluation function). *Let $\mathcal{W} \neq \emptyset$ be some set of premodels, i.e. $\mathcal{W} \subseteq \mathfrak{M}$, and $R_1, \ldots, R_n$ dyadic relations on $\mathcal{W}$. Then $F = \langle \mathcal{W}, R_1, \ldots, R_n \rangle$ is called a modal premodel frame. A function $f : \mathcal{W} \mapsto P(Sent_{\mathcal{L}_{QM}})$ is called an evaluation function relative to F. The set of all evaluation functions relative to a frame F is denoted by Val_F.*

A modal premodel frame together with an evaluation function induce a model for $\mathcal{L}_{QM}$ relative to each member of $\mathcal{W}$.

Definition 4.2 (Models for $\mathcal{L}_{QM}$). *Let F be a frame and $f \in Val_F$ an evaluation function. Then F and f induce a model $M^f = (M, f(M), Y^1_{M}, \ldots, Y^n_{M})$ of the language $\mathcal{L}_{QM}$ relative to every premodel (world) M. $f(M)$ is is the extension of the truth predicate at M and Y^i_{M}, with*

$$Y^i_{M^f} = \bigcap_{M' \in [MR_i]} f(M')$$

the extension of the modal predicates.[13] *$[MR_i]$ is shorthand for the set $\{M' \in \mathcal{W} : MR_iM'\}$.*

Definition 4.3 (Modal Jump relative to a premodel frame). *Let F ba a frame and Val_F be the set of evaluation functions of F. The modal jump relative to F, Ξ_F, is an operation on Val_F such that for all $f \in Val_F$ and all $M \in \mathcal{W}$*

$$[\Xi_F(f)](M) = \{\phi \in Sent_{\mathcal{L}_{QM}} : M^f \models \phi\}$$

Iterative applications of Ξ_F for a given ordinal α are defined by transifinite recursion as in definition 3.4:

$$\Xi^\alpha_F(f) := \begin{cases} f & ,if \quad \alpha = 0 \\ \Xi_F(\Xi^\delta_F(f)) & ,if \quad \alpha = \delta + 1 \\ g \in Val_F & ,if \quad \alpha \text{ is a limit ordinal} \end{cases}$$

where for all $M \in \mathcal{W}$

$$g(M) = \{\phi \in Sent_{\mathcal{L}_{QM}} : \exists\kappa(\phi \in \bigcap_{\kappa \leq \beta < \alpha} [\Xi^\beta_F(f)](M))\}$$

For a given frame F and evaluation function f we sometimes write f^α instead of $\Xi^\alpha_F(f)$.

[12]Since the domains of the premodels may vary we decided to avoid complication and to consider subsets of the set of sentences of $\mathcal{L}_{QM}$ as interpretations of the truth predicate only.

[13]We assume $\bigcap$ to be an operation on $P(Sent_{QM})$ and thus, in particular, $\bigcap \emptyset = Sent_{QM}$.

Thus the iterative application of the modal jump relative to a frame leads us to a sequence of interpretations of the truth predicate and the modal predicates. As before it can be shown that the process reaches a unique fixed-point at the first limit ordinal ω, i.e. lemma 3.5 carries over with minor modification.

Lemma 4.4. *Let F be a modal premodel frame and $f, g \in Val_F$. Then for all $M \in \mathcal{W}$, all natural numbers n and all ordinals α, if $\alpha > n+1$ then for all sentences ϕ with $dq(\phi) \leq n$:*

$$\phi \in [\Xi_F^{n+2}(f)](M) \Leftrightarrow \phi \in [\Xi_F^{\alpha}(g)](M)$$

Proof sketch. We use the construction of the proof of lemma 3.5 and observe that we can construct the mapping σ such that it respects the interpretation of the truth and the modal predicates, i.e. $\sigma : D \longrightarrow D$ is a bijection which is the identity function on the sentences of quotation degree $< n$ and on $D - Sent_{\mathcal{L}_{QM}}$, and additionally,

$$\phi \in [\Xi_F^{n+1}(f)](M) \Leftrightarrow \sigma(\phi) \in [\Xi_F^{\delta}(g)](M)$$
$$\phi \in Y^i_{M^{f^{n+1}}} \Leftrightarrow \sigma(\phi) \in Y^i_{M^{g^{\delta}}}$$

for all i, with $1 \leq i \leq n$. We can find such a function σ due to the following observations: Let U be the interpretation of a modal predicate at a stage γ and V the interpretation of either the truth predicate or one of the remaining modal predicates, then either

- U and V coincide, or
- U is a denumerable subset of V, and $V - U$ is also denumerable, or
- $U \cap V$ is denumerable set of sentences, and so are $U - V$ and $V - U$.[14]

Then given such a function σ the argument used in the proof of lemma 3.5 allows us to conclude to the desired. □

Corollary 4.5. *Let F be a frame. Then for all evaluation functions $f, g \in Val_F$ and all $\phi \in Sent_{\mathcal{L}_{QM}}$*

(*i*) $\Xi_F^{\omega}(f) = \Xi_F^{\omega+1}(g)$

(*ii*) $M^{f^{\omega}} \models \phi \Leftrightarrow \phi \in f^{\omega}(M)$

[14]To see this, note that the interpretations of the truth and modal predicates cannot be disjoint since all the tautologies and the $\mathcal{L}$-truths are in the interpretation of the truth predicate in every model. If there exists a sentence which is not in the interpretation of the one predicate but the other, then there exist denumerable many sentences of this kind: take all the conjunctions of tautologies with this sentence. This explains the second observation. A parallel argument establishes the third.

The corollary implies the existence of unique fixed-points of Ξ_F in Val_F, i.e. there exists exactly one evaluation function $g \in Val_F$ with

$$\Xi_F(g) = g$$

This allows us to define the notion of a proper model of $\mathcal{L}_{QM}$:

Definition 4.6 (Proper Model). *Let F be a frame, g the evaluation function in Val_F with $\Xi_F(g) = g$ and $M \in \mathcal{W}$ a proper premodel. Then the model M^g induced by F and g relative to M is called a proper model of $\mathcal{L}_{QM}$. We write $F, M \models \phi$ to convey the fact that ϕ is true in the proper model induced by the premodel M relative to F.*[15] *If a formula $\phi \in \mathcal{L}_{QM}$ is true in all proper models induced by a frame, we write $F \models \phi$. Let $\mathfrak{F}$ be a class of frames, if for all $F \in \mathfrak{F}(F \models \phi)$ we write $\mathfrak{F} \models \phi$.*

(TB) is true in any modal premodel frame. Moreover, in virtue of our construction the modal principle (K) holds for every modal predicate, i.e.:

Theorem 4.7. *Let $\mathfrak{F}$ be the class of all modal premodel frames. Then for all $\phi, \psi \in Sent_{\mathcal{L}_{QM}}$*

$$\mathfrak{F} \models T\ulcorner\phi\urcorner \leftrightarrow \phi$$
$$\mathfrak{F} \models N_i\ulcorner\phi \rightarrow \psi\urcorner \rightarrow (N_i\ulcorner\phi\urcorner \rightarrow N_i\ulcorner\psi\urcorner)$$

for all i with $1 \leq i \leq n$. Moreover, if ϕ is true in all proper models induced by a frame F, then $N_i\ulcorner\phi\urcorner$ will be true in all models induced by F.

At first sight theorem 4.7 might seem disappointing since our models only satisfy the modal principle (K) but whereas, without doubt, we wish further modal principles to be true. However, this is just parallel to the situation in possible world semantics for modal operator logics. If no assumptions are made on behalf of the accessibility relations of a modal frame, the models based on this frame are only guaranteed to validate the operator versions of (K).

But as in possible world semantics for modal operator logic we can impose conditions on the accessibility relation $R_1, \ldots, R_n$ of a modal premodel frame in order to ensure that further modal principles will be true in a proper model. For example, if we require an accessibility relation R_i of a modal premodel frame F to be reflexive, then for every proper model $\mathcal{M}$ induced by F the principle (T) will turn out true:[16]

$$\mathcal{M} \models N_i\ulcorner\phi\urcorner \rightarrow \phi$$

[15]Note that as a consequence of the uniqueness of fixed-points for every frame F and proper premodel $M \in \mathcal{W}$ there exists only one proper model which is induced by M relative to this frame.

[16]We use calligraphic letters to denote proper models.

Basically, this can be done for all the common, characteristic modal principles including those principles that deal with the interaction of the modal predicates. This highlights how the present semantic approach parallels possible world semantics for modal operator logic to a certain extent. Indeed, we may impose one and the same condition on the accessibility relation of a modal premodel frame and on the accessibility relation of a possible world frame for modal operator logic.

Definition 4.8 (Property Φ). *A modal premodel frame and a modal frame of possible world semantics for modal operator logic can share the conditions imposed on the accessibility relation. We call these conditions "property Φ".*

Comparing the effects of imposing certain properties on the accessibility relation shows that possible world semantics for modal operator logic and the semantics we just developped work pretty much in the same way. In general, if a certain property Φ is imposed on the accessibility relations of a modal premodel frame and a modal frame for modal operator logic respectively, the "same" modal principles will be true in both frames. To use our example once more: if the accessibility relation is reflexive, the predicate version and the operator version of the modal principle (T) will be true in the modal premodel frame or, respectively, the modal frame, i.e. the possible world frame of modal operator logic.

We can exploit the structural similarities between the two semantics in order to show that the predicate approach to modalities is adequate with respect to a wide class of multimodal operator logics. Moreover, the semantics that we have developed is rather flexible and intuitive to the extent possible world semantics for modal operator logic is. Accordingly, the champion of possible world semantics should be tempted by the present approach.

5 Adequacy of the Predicate Approach to Modalities

The idea behind the adequacy result is to show that the syntactical approach validates exactly the theorems of modal operator logic. Of course, it remains to be specified what it means for the syntactical approach to validate the same theorems as we are dealing with two different languages and as a trivial consequence a theorem of modal operator logic cannot be true in the syntactical approach (nor can it be false).

Still the gist of the affirmation seems to be pretty clear. If a modal operator logic proves a certain modal statement, then the corresponding modal statement of the quotation name language, i.e. the statement in which the modal operators of the initial statement are now read as modal predicates and the formula in the scope of the operator will be placed between quotation marks in order to form a quotation name, should be true in an appropriate modal premodel frame. The idea is to translate the propositional atoms of the modal operator language into sentences of the language $\mathcal{L}_{QM}$ and to guarantee that the translation function, say H, commutes with the boolean connectives and translates a formula $\Box_i\phi$ as $N_i\langle H(\phi)\rangle$. To this end let $\mathcal{L}_{\blacksquare\Box}$ be a multimodal language whose syntax is given by

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid \blacksquare\phi \mid \Box_i\phi$$

for $1 \leq i \leq n \in \omega$ and $p \in At_{\blacksquare\Box}$ where $At_{\blacksquare\Box}$ denotes the set of propositional atoms of the language. $\blacksquare$ is meant to be a truth operator. We define the notion of a translation H^* over a realization $*$.

Definition 5.1 (Translation). *A mapping $* : At_{\blacksquare\Box} \longrightarrow Sent_{\mathcal{L}_{QM}}$ is called a realization. H^* is a translation function from $\mathcal{L}_{\blacksquare\Box}$ into $\mathcal{L}_{QM}$ iff it respects the following conditions:*

$$H^*(\phi) := \begin{cases} \phi^* & , if \quad \phi \in At_{\blacksquare\Box_i} \\ \bot & , if \quad \phi \doteq \bot \\ \neg H^*(\psi) & , if \quad \phi \doteq (\neg\psi) \\ H^*(\psi) \wedge H^*(\chi) & , if \quad \phi \doteq (\psi \wedge \chi) \\ T\langle H^*(\psi)\rangle & , if \quad \phi \doteq (\blacksquare\psi) \\ N_i\langle H^*(\psi)\rangle & , if \quad \phi \doteq (\Box_i\psi), \text{ for } 1 \leq i \leq n \end{cases}$$

Using this notion of translation we can spell out our adequacy condition more precisely. A sentence of the modal operator language is a theorem of a modal operator logic under consideration, if and only if for every realization its translation will be true in an appropriate class of modal premodel frames of the syntactical approach.

The class of modal premodel frames appropriate with respect to a modal operator logic will be determined with respect to property Φ: if a modal operator logic is complete with respect to the class of modal frames with property Φ, the class of modal premodel frames with property Φ will be considered as appropriate.

Theorem 5.2. *Let $\mathcal{F}_{\blacksquare\Box}$ be the class of possible world frames with property Φ and $\mathfrak{F}$ the class of modal premodel frames with property Φ. Then for all $\phi \in \mathcal{L}_{\blacksquare\Box}$*

$$\mathcal{F}_{\blacksquare\Box} \models \phi \Leftrightarrow \textit{for all realizations } * \, (\mathfrak{F} \models H^*(\phi))$$

To establish theorem 5.2 we shall employ two lemmata. The first one shows us how to construct a possible world frame starting from a proper premodel frame whereas the second establishes the converse direction:

Lemma 5.3. *For all modal premodel frames F with property Φ, there exists a possible world frame $F_{\blacksquare\Box}$ with property Φ such that for all $\phi \in \mathcal{L}_{\blacksquare\Box}$:*

$$F_{\blacksquare\Box} \models \phi \Leftrightarrow \textit{for all realizations } * \, (F \models H^*(\phi))$$

Proof. Let $F = \langle \mathcal{W}, R_1, \ldots, R_n \rangle$ be the modal premodel frame. Now, take $\mathcal{W}$ to be the set W of worlds of the possible world frame of modal operator logic and define $R_{\Box_i} := R_i$. Set $F_{\blacksquare\Box} := \langle W, R_1, \ldots, R_n \rangle$. We may verify by an induction over the complexity of ϕ that for all $M \in \mathcal{W}$

$$F_{\blacksquare\Box}, M \models \phi \Leftrightarrow \forall * (F, M \models H^*(\phi))$$

We discuss $\phi \doteq \Box_i \psi$ and leave the rest to the reader. To this end assume $F_{\blacksquare\Box}, M \models \Box_i \psi$. By definition the latter is equivalent to

$$\text{for all valuations } V\, \forall M'(MR_{\Box_i}M' \Rightarrow (F_{\blacksquare\Box_i}, V), M' \models \psi).$$

Since V does not occur in the antecedent this is equivalent to

$$\forall M'(MR_{\Box_i}M' \Rightarrow \forall V((F_{\blacksquare\Box}, V), M' \models \psi)).$$

By induction hypothesis and definition we get

$$\forall M'(MR_iM' \Rightarrow \forall * (F, M' \models H^*(\psi)))$$

and since $*$ is not bound in the antecedent of the above conditional we may conclude to the desired. $\Box$

Lemma 5.4. *For all possible world frames $F_{\blacksquare\Box}$ with property Φ, there exists a modal premodel frame F with property Φ such that for all $\phi \in \mathcal{L}_{\blacksquare\Box}$:*

$$F_{\blacksquare\Box} \models \phi \Leftrightarrow \textit{for all realizations } * (F \models H^*(\phi))$$

Proof. Let $F_{\blacksquare\Box}$ be the frame $\langle W, R_{\Box_1}, \ldots, R_{\Box_n} \rangle$. Take some set of proper premodels A with $|W| \leq |A|$ and let $\gamma : W \longrightarrow A$ be some injective mapping. We define

$$\mathcal{W} := \{\gamma(w) : w \in W\}$$
$$R_i = \{\langle \gamma(w), \gamma(v) \rangle : wR_{\Box_i}v\}$$

and set $F = \langle \mathcal{W}, R_1, \ldots, R_n \rangle$. Again we may then verify by induction on the complexity of ϕ that

$$F_{\blacksquare\Box}, w \models \phi \Leftrightarrow \forall * (F, \gamma(M) \models H^*(\phi))$$

which establishes the lemma. $\Box$

We may now state the proof of the main theorem as a trivial consequence of the two lemmata:

Proof of theorem 5.2. Suppose $\mathcal{F}_{\blacksquare\square} \models \phi$ but that there exists a modal premodel frame F with property Φ and $\neg\forall * (F \models H^*(\phi))$. Then by lemma 5.3 we end up in contradiction. On the other hand assuming $\forall * (\mathfrak{F} \models H^*(\phi))$ and $\mathcal{F}_{\blacksquare\square} \not\models \phi$ equally leads to contradiction by lemma 5.4. $\square$

We get a nice corollary from this result which can be read as the exact formal rendering of the adequacy criterium we laid out:

Corollary 5.5. *Let $\mathcal{S}$ be a modal operator logic complete with respect to the class of possible world frames $\mathcal{F}$ with property Φ. Let $\mathfrak{F}$ be the class of modal premodel frames with property Φ, then for all $\phi \in \mathcal{L}_{\blacksquare\square}$:*

$$\mathcal{S} \vdash \phi \Leftrightarrow \forall * (\mathfrak{F} \models H^*(\phi))$$

Proof. By completeness of $\mathcal{S}$ we have for all $\phi \in \mathcal{L}_{\blacksquare\square_i}$

$$\mathcal{S} \vdash \phi \Leftrightarrow \mathcal{F} \models \phi$$

which together with theorem 5.2 establishes the claim. $\square$

As we have argued the two results, theorem 5.2 and corollary 5.5 may be taken to show the adequacy of the predicate approach to modality from the perspective of modal operator logic. With this observation we end the present section and turn toward the conclusion of our investigation.

6 Conclusion

The syntactical approach to modality we have just outlined appears to vindicate predicate approaches to modality from the perspective of modal operator logic. As we have seen the approach allows us to adhere to the unrestricted modal principles but at the same time proves to be adequate from the perspective of modal operator logic. Moreover, the approach comes with a semantics which is intuitive to the extent possible world semantics for the modal operator can be considered to be intuitive. And even though the approach we have outlined so far does not come with a developed proof theory the result of Asher and Kamp [2], who provide a complete axiomatization of their semantics in the absence of vicious forms of self-reference, suggest that this can be done along the lines of Niemi [14].

While the foregoing establishes that we may conceive of modalities as predicates we have not provided an argument to the effect that we should. It is often thought that one advantage of the predicate approach as opposed to the operator treatment lies in its greater expressive strength which allows us to provide a more adequate formalization of modal discourse from the perspective of natural language. And indeed it is unclear whether the present proposal is satisfactory from this latter perspective as for obvious reasons we may not, e.g. express modal self-reference or other natural language phenomena which require the modal theory to speak about its own syntax. Accordingly, a common reaction to vindications of the predicate approach which embrace the unrestricted modal principles that in other settings would lead to paradox has been to question whether these approaches can be considered as predicate approaches to modality proper.[17] We think, however, that this kind of argument is beside the point as the modal notions have been introduced as predicates into the language and no syntactic restrictions whatsoever have been placed on the formation rules of the language. So there is no reason why they should be considered as operators.

We think the moral to be drawn is somewhat different. Montague's theorem should not be used as an argument against the predicate approach to modality but rather it should be understood as a limitative result. That is, if we wish to have an expressively rich framework, i.e. a framework in which modal self-reference is expressible, then independently of which category of logical grammar we assume to aptly formalize the modal notion, we need to give up the unrestricted modal principles we know from standard modal operator logic. If we are happy with an expressively weak framework, then again independently of the grammatical category assumed we may uphold the standard laws of modal logic.[18]

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[17] The argument which appears at least implicitly in the literature can be glossed as follows. The fact that you have shown that modal logic, i.e. the principles of modal operator logic, can be upheld in a predicate setting just shows that the modal predicates under consideration are just operators in disguise. This kind of argument is maybe most explicitly stated in the writings of Reinhard [15] and Grim [4].

[18] See Stern [19] for a more in depth discussion of the distinction between modal operators and predicates, and arguments in favor of expressively rich frameworks for modality.

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